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A Weierstrass-Like Theorem for Real Separable Hilbert Spaces*

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1. INTRODUCTION

In [2–7] several versions of the Weierstrass theorem concerning approximation of continuous functions by means of polynomials in the case of Hilbert and Banach spaces are given. In all of these papers a crucial role is played by the compactness of the domain. Such a hypothesis appears to be quite restrictive in some important applications in which only boundedness of the domain is available: e.g., those regarding the approximation of input–output maps by means of Volterra series expansion in System Theory [1].

In this paper the boundedness of the domain will be assumed; however, a stronger continuity property of the function is required in order to get the polynomial approximation sought. Such a property appears to be a very "physical" one in System Theory, in that it corresponds to a smoothing action of the input–output map.

2. PRELIMINARIES

Let $\mathscr{H}$, $\mathscr{H}'$ be real separable Hilbert spaces and $\mathscr{B}$, $\mathscr{B}'$ Banach spaces. Let us recall that a n-linear operator M on $\mathscr{H}^n \triangleq \mathscr{H} \times \cdots \times \mathscr{H}$ (n times) into $\mathscr{H}'$ is a map which is linear in each of its arguments separately. It is natural to define

$$M_n x^n \triangleq \underbrace{M(x,\dots,x)}_{n\ \text{times}}$$

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as the n-degree monomial operator (associated to the multilinear operator M) on $\mathscr{H}$ into $\mathscr{H}'$. Then the operator $P: \mathscr{H} \to \mathscr{H}'$, defined as

$$P(x) \triangleq M_n x^n + M_{n-1} x^{n-1} + \cdots + M_1 x + M_0,$$

where M_0 stands for a constant operator, is called an n-degree polynomial operator.

We need also to recall the following definition:

DEFINITION. A map $F: \mathscr{H} \to \mathscr{B}$ is said to be uniformly continuous with respect to the S-topology (by Sazonov [8]) if, for any $\varepsilon > 0$, there exists a self-adjoint nonnegative definite trace-class linear operator $S_\varepsilon: \mathscr{H} \to \mathscr{H}$ such that

$$[S_\varepsilon(x' - x''), x' - x''] < 1; \qquad x', x'' \text{ in } \mathscr{H} \tag{1}$$

implies

$$\| F(x') - F(x'')\| < \varepsilon. \tag{2}$$

Remark. In the linear Hilbert space case the uniform S-continuity implies the Hilbert–Schmidt property for F. Note that the S-topology is weaker than the norm topology.

The following results may be useful to characterize the class of uniformly S-continuous functions on bounded domains.

THEOREM 1. *Any map $F: \mathscr{H} \to \mathscr{B}$ which is uniformly continuous with respect to the S-topology is also compact.*

Proof. We will have to show that F carries bounded sets in $\mathscr{H}$ into sets whose closure is compact. Let Ω be any bounded set in $\mathscr{H}$ and let $\bar{\Omega}$ be its closure. Then $\overline{F(\Omega)}$ is compact. For, let $\{y_n\}$ be any sequence in $F(\Omega)$, then the sequence $\{x_n \in \Omega: y_n = F(x_n)\}$ admits a subsequence $\{x_{n_k}\}$ weakly convergent by the weak compactness theorem. Thus for any Hilbert–Schmidt operator $L: \mathscr{H} \to \mathscr{H}$ (actually any compact operator), the sequence $\{Lx_{n_k}\}$ is strongly convergent and the same holds for the sequence $\{F(x_{n_k})\}$ because of the uniform S-continuity. ∎

THEOREM 2. *Let $G: \mathscr{H} \to \mathscr{B}'$ be a map uniformly continuous with respect to the S-topology on a bounded set Ω in $\mathscr{H}$, and $g: \mathscr{B}' \to \mathscr{B}$ a continuous map. Then the map $F = g \circ G$ is uniformly continuous with respect to the S-topology on Ω.*

Proof. The result is readily achieved by observing that

$$\| F(x) - F(x')\| = \| g(G(x)) - g(G(x'))\| < \varepsilon; \qquad x, x' \text{ in } \Omega \tag{3}$$

if

$$\| G(x) - G(x')\| < \delta_\epsilon \tag{4}$$

because g is uniformly continuous on $G(\Omega)$, $\overline{G(\Omega)}$ being compact by Theorem 1. Moreover, from the uniform S-continuity of G, there exists a self-adjoint nonnegative definite trace-class operator $S_{\delta_\epsilon}: \mathscr{H} \to \mathscr{H}$ such that, for any x, x' in Ω satisfying

$$[S_{\delta_\epsilon}(x - x'), x - x'] < 1, \tag{5}$$

inequalities (4), and then (3), hold. ∎

3. Approximation in a Bounded Domain

Now we can state the main result:

THEOREM 3. *Let $F: \mathscr{H} \to \mathscr{H}'$ be continuous, and let Ω be any bounded set in $\mathscr{H}$. If F restricted to Ω is uniformly continuous with respect to the S-topology then, for any $\varepsilon > 0$, there exists a continuous polynomial P on $\mathscr{H}$ into $\mathscr{H}'$ such that*

$$\sup_{x \in \Omega} \| F(x) - P(x)\| < \varepsilon. \tag{6}$$

Proof. The proof goes along the same lines as in Prenter's theorem [6]. Let $\{\varphi_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ be orthonormal bases for $\mathscr{H}$ and $\mathscr{H}'$ respectively and $\Pi^n_{\mathscr{H}}$, $\Pi^m_{\mathscr{H}'}$ the projection operators from $\mathscr{H}$ and $\mathscr{H}'$ onto $H_n \triangleq \text{span}\{\varphi_1, ..., \varphi_n\}$ and $H'_m \triangleq \text{span}\{\psi_1, ..., \psi_m\}$. So we can define the map $F^n_m: \mathscr{H} \to \mathscr{H}'$ as

$$F^n_m \triangleq \Pi^m_{\mathscr{H}'} \circ F \circ \Pi^n_{\mathscr{H}}.$$

Observe now that

$$\sup_{x \in \Omega} \| F(x) - P(x)\|$$
$$\leqslant \sup_{x \in \bar{\Omega}} \| F(x) - P(x)\| \leqslant \sup_{x \in \bar{\Omega}} \|F(x) - F^n_m(x)\| + \sup_{x \in \bar{\Omega}} \|F^n_m(x) - P(x)\|. \tag{7}$$

We start proving that for any n, m, given $\varepsilon > 0$, there exists a continuous polynomial P such that

$$\sup_{x \in \bar{\Omega}} \| F^n_m(x) - P(x)\| < \varepsilon/2. \tag{8}$$

For, let $K_n \triangleq \Pi^n_{\mathscr{Y}}(\bar{\Omega})$. K_n is compact. Let then $\tilde{F}^n_m$ be the restriction of F^n_m to H_n. $\tilde{F}^n_m$ is a continuous map from K_n into H'_m and, as such, by a slight modification of Theorem 4.1 in]6], can be uniformly approximated in K_n by a continuous polynomial $\tilde{P}:H_n \to H'_m$, so that

$$\sup_{x \in K_n} \|\tilde{F}^n_m(x) - \tilde{P}(x)\| < \varepsilon/2.$$

Now let us extend $\tilde{P}$ to P defined on the whole $\mathscr{H}$ as follows

$$P(x) \triangleq \tilde{P} \circ \Pi^n_{\mathscr{Y}} x, \qquad x \in \mathscr{H}.$$

Clearly P is a continuous polynomial of finite rank on $\mathscr{H}$. Thus

$$\begin{aligned}
&\sup_{x \in \bar{\Omega}} \|F^n_m(x) - P(x)\| \\
&\quad = \sup_{x \in \bar{\Omega}} \|\Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} x - \tilde{P} \circ \Pi^n_{\mathscr{Y}} x\| \\
&\quad = \sup_{x \in \bar{\Omega}} \|\Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} \circ \Pi^n_{\mathscr{Y}} x - \tilde{P} \circ \Pi^n_{\mathscr{Y}} x\| \\
&\quad = \sup_{z \in K_n} \|\Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} z - \tilde{P}(z)\| \\
&\quad = \sup_{z \in K_n} \|\tilde{F}^n_m(z) - \tilde{P}(z)\| < \varepsilon/2.
\end{aligned}$$

Now it remains to prove that

$$\sup_{x \in \bar{\Omega}} \|F(x) - F^n_m(x)\| < \varepsilon/2 \tag{9}$$

for n and m enough large. For, we observe that

$$\begin{aligned}
\sup_{x \in \bar{\Omega}} \|F(x) - F^n_m(x)\| &= \sup_{x \in \bar{\Omega}} \|F(x) - \Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} x\| \\
&\leqslant \sup_{x \in \bar{\Omega}} \|F(x) - F \circ \Pi^n_{\mathscr{Y}} x\| \\
&\qquad + \sup_{x \in \bar{\Omega}} \|F \circ \Pi^n_{\mathscr{Y}} x - \Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} x\|.
\end{aligned} \tag{10}$$

Let us consider the compact set $C_n \triangleq F \circ \Pi^n_{\mathscr{Y}}(\bar{\Omega})$. Then, by the Lemma 5.2 in [6], given $\varepsilon > 0$ and any integer n, there exists a $\mu(\varepsilon, n)$ such that

$$\begin{aligned}
&\sup_{x \in \bar{\Omega}} \|F \circ \Pi^n_{\mathscr{Y}} x - \Pi^m_{\mathscr{Y}'} \circ F \circ \Pi^n_{\mathscr{Y}} x\| \\
&\quad = \sup_{y \in C_n} \|y - \Pi^m_{\mathscr{Y}'} y\| < \varepsilon/4, \ \forall m > \mu(\varepsilon, n).
\end{aligned} \tag{11}$$

Now, (9) follows from (10) and (11) as soon as we show the existence of a $\nu(\varepsilon)$ such that

$$\sup_{x\in\bar{\Omega}} \|F(x) - F \circ \Pi^n_{\mathscr{V}} x\| \leqslant \varepsilon/4, \qquad \forall n > \nu(\varepsilon). \tag{12}$$

From the definition of uniform continuity with respect to the S-topology, it follows that, given $\varepsilon > 0$, there exists a self-adjoint nonnegative definite trace-class operator $S_{\varepsilon/4}: \mathscr{H} \to \mathscr{H}$, such that

$$\|F(x) - F \circ \Pi^n_{\mathscr{V}} x\| \leqslant \varepsilon/4; \qquad x \text{ in } \bar{\Omega} \tag{13}$$

if

$$[S_{\varepsilon/4}(x - \Pi^n_{\mathscr{V}} x), x - \Pi^n_{\mathscr{V}} x] < 1. \tag{14}$$

So we will conclude the proof by showing that (14) is uniformly verified with respect to $x \in \bar{\Omega}$ for n enough large. For, denoting by $L_\varepsilon: \mathscr{H} \to \mathscr{H}$, the Hilbert–Schmidt operator, such that $S_{\varepsilon/4} = L^*_\varepsilon L_\varepsilon$, we have

$$\begin{aligned}
&\sup_{x\in\bar{\Omega}} [S_{\varepsilon/4}(x - \Pi^n_{\mathscr{V}} x), x - \Pi^n_{\mathscr{V}} x] \\
&\quad = \sup_{x\in\bar{\Omega}} [L^*_\varepsilon L_\varepsilon (x - \Pi^n_{\mathscr{V}} x), x - \Pi^n_{\mathscr{V}} x] \\
&\quad = \sup_{x\in\bar{\Omega}} \left[L_\varepsilon \sum_{i=n+1}^{\infty} [x, \varphi_1]\, \varphi_i, L_\varepsilon \sum_{j=n+1}^{\infty} [x, \varphi_j]\, \varphi_j\right] \\
&\quad = \sup_{x\in\bar{\Omega}} \left\| \sum_{i=n+1}^{\infty} [x, \varphi_i]\, L_\varepsilon \varphi_i \right\|^2 \\
&\quad \leqslant \sup_{x\in\bar{\Omega}} \left\{ \sum_{i=n+1}^{\infty} |[x, \varphi_i]| \cdot \|L_\varepsilon \varphi_i\| \right\}^2 \\
&\quad \leqslant \sup_{x\in\bar{\Omega}} \sum_{i=n+1}^{\infty} [x, \varphi_i]^2 \sum_{i=n+1}^{\infty} \|L_\varepsilon \varphi_i\|^2 \\
&\quad \leqslant \sup_{x\in\bar{\Omega}} \|x\|^2 \sum_{i=n+1}^{\infty} \|L_\varepsilon \varphi_i\|^2 \\
&\quad = M^2 \sum_{i=n+1}^{\infty} \|L_\varepsilon \varphi_i\|^2,
\end{aligned} \tag{15}$$

where $M \triangleq \sup_{x\in\bar{\Omega}} \|x\|$. This implies that an integer $\nu(\varepsilon)$ exists such that (14) holds for $n > \nu(\varepsilon)$; because (13) is satisfied for any $x \in \bar{\Omega}$, (9) follows. ∎

Remark. Note that Theorem 3 is equivalent to the denseness of the continuous polynomials with respect to the family of uniformly S-continuous functions.

References

1. A. Bertuzzi, A. Gandolfi, and A. Germani, Causal polynomial approximation for input–output maps on Hilbert spaces, *Math. Systems Theory*, in press.
2. M. Chaika and S. J. Perlman, A Weierstrass theorem for a complex separable Hilbert space, *J. Approximation Theory* **15** (1975), 18–22.
3. P. G. Gallman and K. S. Narendra, Representations of nonlinear systems via the Stone–Weierstrass theorem, *Automatica* **12** (1976), 619–622.
4. V. I. Istraţescu, A Weierstrass theorem for real Banach spaces, *J. Approximation Theory* **19** (1977), 118–122.
5. W. A. Porter and T. M. Clark, Causality structure and the Weierstrass theorem, *J. Math. Anal. Appl.* **52** (1975), 351–363.
6. P. M. Prenter, A Weierstrass theorem for real, separable Hilbert spaces, *J. Approximation Theory* **3** (1970), 341–351.
7. W. L. Root, On the modeling of systems for identification. Part I: ε-representations of classes of systems, *SIAM J. Contr.* **13** (1975), 927–944.
8. V. V. Sazonov, A remark on characteristic functionals, *Teor. Veroj. Prim.* **3** (1958), 201–205.